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CONTRIBUTIONS TO THE THEORY OF RANK ORDER STATISTICS THE TWO SAMPLE CASE: FINE STRUCTURE OF THE ORDERING OF PROBABILITIES OF RANK ORDERS*

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CONTRIBUTIONS TO THE THEORY OF RANK ORDER STATISTICS—THE TWO SAMPLE CASE: FINE STRUCTURE OF THE ORDERING OF PROBABILITIES OF RANK ORDERS

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1. Introduction.

In constructing admissible two sample rank order tests one needs information on the ordering of probabilities of rank orders. Specifically, if, under some restriction of the class of alternatives, the rejection region of a test contains the rank order z then it should contain all rank orders more probable than z.

This paper contains several theorems on such orderings under various alternatives, especially the location parameter case for symmetric distributions.

2. Notation and Assumptions.

$X = (X_1,\ldots,X_m)$ and $Y = (Y_1,\ldots,Y_n)$ are samples drawn from absolutely continuous populations with densities $f(\cdot)$ and $g(\cdot)$, respectively. $F(\cdot)$ and $G(\cdot)$ denote the corresponding distributions.

$W = (W_1,\ldots,W_{m+n})$ denotes the order statistics of the combined sample, $(X, Y) = (X_1,\ldots,X_m, Y_1,\ldots,Y_n)$, and $Z = (Z_1,\ldots,Z_{m+n})$ is a random vector of zeros and ones whose $i^{\underline{th}}$ component, Z_i, is 0 if W_i comes from $f(\cdot)$ and 1 if W_i comes from $g(\cdot)$.

Let $z = (z_1,\ldots,z_{m+n})$ be a fixed vector of zeros and ones; we define the complement of z, $z^c = (z_1^c,\ldots,z_{m+n}^c)$ and the transpose of z, $z^t = (z_1^t,\ldots,z_{m+n}^t)$, to be the vectors whose $i^{\underline{th}}$ components are $1-z_i$ and $z_{m+n+1-i}$, respectively. $P(z) = \Pr\{Z=z\}$ denotes the probability of the rank order z.

Since the following restrictions of f and g are assumed in several results below, we list them now along with a shorthand notation.

Restrictions:

ST: $f(x) = f(-x)$ and $g(x) = f(x-\theta)$, where θ is a non-negative constant.

U: $f(x) \geq f(x')$ if $0 \leq x < x'$ or $x' < x \leq 0$.

MLR: $g(y)/f(y) \geq g(x)/f(x)$ if $x \leq y$.

N: $f(\cdot)$ and $g(\cdot)$ are normal densities with common variance 1 and means 0 and θ, respectively, where $\theta \geq 0$.

Note: ST stands for Symmetry and Translation and U implies that $f(\cdot)$ is Unimodal. It is assumed, without loss of generality, that the mode of $f(\cdot)$ is the origin. MLR stands for Monotone Likelihood Ratio and N stands for Normality. Of course N is the strongest and implies the other three. Under ST and/or N we use the notations $P(\mathbf{z})$ and $P(\mathbf{z}|\theta)$ interchangeably.

3. Theorems on the Ordering of Rank Order Probabilities.

The general expression for $P(\mathbf{z})$ is

$$(3.1) \qquad P(\mathbf{z}) = m!n! \int \cdots \int \prod_{i=1}^{m+n} h_{z_i}(t_i)dt_i ,$$

where $h_{z_i}(t_i) = \begin{cases} f(t_i) & z_i=0 \\ g(t_i) & z_i=1 \end{cases}$, and the region of integration is $-\infty < t_1 \leq \ldots \leq t_{m+n} < \infty$. In particular, under ST

$$(3.2) \qquad h_{z_i}(t_i) = f(t_i-\theta z_i).$$

Theorem 1:

If ST holds, then for all θ

i) $P(\mathbf{z}|\theta) = P(\mathbf{z}^t|-\theta)$ and

ii) $P(\mathbf{z}|\theta) = P(\mathbf{z}^c|-\theta)$.

Proof:

Recall the definition of z^t and z^c and note that $f(x) = f(-x)$. In the integral (3.1) (using (3.2)) make the transformation

$$\text{i}')\quad t_i = -t'_{m+n+1-i} \qquad (i=1,2,\ldots,m+n)$$

or

$$\text{ii}')\quad t_i = \theta+t'_i \qquad (i=1,2,\ldots,m+n)$$

and i) or ii) follows at once.

Theorem 2:

If ST holds, then for all θ

$$P(z|\theta) = P(z^{tc}|\theta). \qquad \text{(See Savage (1957) p. 975.)}$$

Proof:

Note that $z^{tc} = (z^t)^c$. Thus, by Theorem 1,

$$P(z^{tc}|\theta) = P(z^t|-\theta) = P(z|\theta).$$

Remark

If a result of the form $P(z|\theta) \geqq P(z'|\theta)$ for $\theta \in \Gamma$ is true under ST, then, by Theorem 2, the following are also true when $\theta \in \Gamma$:

$$P(z^{tc}|\theta) \geqq P(z'|\theta)$$

$$P(z^{tc}|\theta) \geqq P((z')^{tc}|\theta)$$

$$P(z|\theta) \geqq P((z')^{tc}|\theta) .$$

Theorem 3:

If MLR holds and z and z' differ only in their $i^{\underline{th}}$ and $j^{\underline{th}}$ components $(i < j)$ with $(z_i, z_j) = (0, 1)$ while $(z'_i, z'_j) = (1, 0)$, then $P(z) \geqq P(z')$. (See Savage (1956) p. 594.)

Remark

If z and z' have a common number of zeros and ones and are such that $\sum_{j=1}^{i} (z'_j - z_j) \geq 0$, for $i=1,\ldots,m+n$, then there exist $z^1, z^2, \ldots, z^p$, where $z^k = (z^k_1, z^k_2, \ldots, z^k_{m+n})$ for $k=1,\ldots,p$ and $z^1 = z'$, $z^p = z$, such that for $k=2,\ldots,p$, z^{k-1} and z^k differ in exactly two components, i_k and j_k $(i_k < j_k)$ with $(z^k_{i_k}, z^k_{j_k}) = (0, 1)$ and $(z^{k-1}_{i_k}, z^{k-1}_{j_k}) = (1, 0)$.

For example:

$$\begin{aligned} z = z^4 &= (0,0,1,0,1,0,1) \\ z^3 &= (0,0,1,0,1,1,0) \\ z^2 &= (0,0,1,1,0,1,0) \\ z' = z^1 &= (0,1,0,1,0,1,0) \ . \end{aligned}$$

Therefore we have the following result.

Corollary:

If MLR holds and z and z' have the same number of zeros and ones and are such that $\sum_{j=1}^{i} (z'_j - z_j) \geq 0$, for $i=1,\ldots,m+n$, then $P(z) \geq P(z')$. The properties of the orderings implied by Theorem 3 are discussed in detail in Savage (1962).

In succeeding pages we employ the notation (z, w), where $w = (w_1,\ldots,w_{p+q})$ and $z = (z_1,\ldots,z_{m+n})$, to denote the combined vector $(z_1,\ldots,z_{m+n}, w_1,\ldots,w_{p+q})$.

Theorem 4:

If ST and U hold, $\theta \geq 0$, and z contains the same number, r, of zeros and ones, then

i) $P(0,0,1,z^{tc}) \geq P(z,0,0,1)$

and

ii) $P(1,0,0,z^{tc}) \geq P(z,1,0,0)$.

Proof:

i) By Theorem 1, a necessary and sufficient condition for the conclusion is that

$$P(z,0,1,1) \geq P(z,0,0,1) ,$$

which, by (3.1) and (3.2) is equivalent to the inequality

$$(3.3) \qquad I = \int_{-\infty}^{\infty} H(x)\, F(\theta-x)[f(x-\theta)-f(x)]dx \;\geq\; 0, \qquad \text{for all } \theta \geq 0,$$

where $H(x) = (r+2)!(r+1)! \int \cdot \cdot \cdot \int_{-\infty < t_1 \leq \ldots \leq t_{2r+1} \leq x} \prod_{i=1}^{2r} f(t_i - z_i\theta)dt_i \; f(t_{2r+1})dt_{2r+1}$.

We note here for future use that

$$(3.4) \qquad H'(x) = \frac{d}{dx} H(x)$$

$$= f(x)(r+1)!(r+2)! \int \cdot \cdot \cdot \int_{-\infty < t_1 \leq \ldots \leq t_{2r} \leq x} \prod_{i=1}^{2r} f(t_i - \theta z_i)dt_i$$

$$= f(x)\; G(x), \text{ say.}$$

Let $i(x)$ denote the integrand in (3.3) and let

$$I_1 = \int_{\theta/2}^{\infty} i(x)dx , \qquad I_2 = \int_{-\infty}^{\theta/2} i(x)dx .$$

In I_2 make the change of variable $x = \theta - x'$. This yields, after replacing x' by x in the transformed I_2 and adding I_1 and I_2,

$$I = \int_{\theta/2}^{\infty} \left[\frac{H(x)}{F(x)} - \frac{H(\theta-x)}{F(\theta-x)}\right] F(x)\, F(\theta-x)[f(x-\theta)-f(x)]dx .$$

It is easily seen that STU implies $[f(x-\theta)-f(x)] \geqq 0$ for $x \geqq \theta/2$. Therefore, a sufficient condition for I to be non-negative is that

$$\frac{H(x)}{F(x)} \geqq \frac{H(\theta-x)}{F(\theta-x)} \quad \text{for } x \geqq \theta/2.$$

Since $x \geqq \theta/2$ implies $x \geqq \theta-x$ it suffices to show that $\frac{H(x)}{F(x)}$ is non-decreasing for all x. And for this to be true it is sufficient that $\frac{H(x)}{F(x)}$ has a non-negative derivative, i.e., that

$$H'(x)F(x)-H(x)f(x) \geqq 0$$

or, by (3.4), that

$$G(x)F(x)-H(x) \geqq 0.$$

Since $G(x)F(x)-H(x) = 0$ at $x = -\infty$ it suffices to show that $G(x)F(x)-H(x)$ is non-decreasing for all x, or that

$$G'(x)F(x)+f(x)G(x)-H'(x) \geqq 0$$

or, by (3.4), that

$$G'(x) \geqq 0,$$

which is clearly so.

ii) The proof is identical with that of i) with the following trivial modification: (3.3) is replaced by

$$I = \int_{-\infty}^{\infty} H_1(x)F(-x)[f(x-\theta)-f(x)]dx \geqq 0, \qquad \text{for all } \theta \geqq 0,$$

where $H(x) = (r+2)!(r+1)! \int \cdot \cdot \cdot \int_{-\infty < t_1 \leqq \ldots \leqq t_{2r+1} \leqq x} \prod_{i=1}^{2r} f(t_i-s_i\theta)dt_i \, f(t_{2r+1}-\theta)dt_{2r+1}$.

The proofs of the next three theorems have several features in common which

we note here. They all state that if N holds then $P(z) \geqq P(z')$. Equivalent conclusions are $P(z^{tc}) \geqq P(z')$ and $P(z) \geqq P(z'^{tc})$; one or the other is noted in each theorem and is in fact what is proved.

The first step is to replace each $P(\cdot)$ with its equivalent under (3.1) and (3.2) and to change the order of integration so that a particular pair of variables is integrated last. For the convenience of the reader this pair of variables will be indicated by adding primes (') to the corresponding entries in the z-vectors the first time they appear.

At this point we have an inequality of the form

$$\int_{-\infty}^{\infty} \int_{x}^{\infty} S(x, y)dydx \geqq 0$$

as a necessary and sufficient condition for the inequality $P(z) \geqq P(z')$. By making the transformation $y-x' = w$, $x = x'$ we obtain the equivalent inequality (omitting primes)

$$\int_{0}^{\infty} \int_{-\infty}^{\infty} S(x, x+w)dxdw \geqq 0.$$

A sufficient condition for this inequality is that the inner integral is non-negative for $w \geqq 0$, i.e., that

$$I(w) = \int_{-\infty}^{\infty} S(x, x+w)dx \geqq 0,$$

for $w \geqq 0$.

Let $I_1(w) = \int_{-\infty}^{(\theta-w)/2} S(x, x+w)dx$ and $I_2(w) = I(w)-I_1(w)$. In $I_1(w)$ make the transformation $x' = \theta-x-w$; this makes the ranges of integration of

$I_1(w)$ and $I_2(w)$ coincide.

By adding $I_1(w)$ and $I_2(w)$, we obtain

$$I(w) = \int_{(\theta-w)/2}^{\infty} T(x, w)dx ,$$

where $T(x, w) = [S(x; x+w)+S(\theta-x-w; \theta-x)]$.

In each case we show that $T(x, w) \geq 0$ for $x \geq (\theta-w)/2$ and $w \geq 0$, which, of course, implies $I(w) \geq 0$ for $w \geq 0$.

In the proof of Theorem 5 we shall repeat in detail the argument just outlined. By Theorem 1.i it is necessary to consider only $\theta > 0$ in proving Theorems 5 and 6.

Theorem 5:

If N holds and $\theta \neq 0$, then $P(1,0,0^r,0,1) > P(0,1,0^r,1,0)$ or, equivalently, $P(1,0',0^r,0',1) > P(1,0',1^r,0',1)$. ($x^r$ denotes a vector of r x's.)

Proof:

By (3.1) and (3.2), the inequality

$$P(1,0,0^r,0,1)-P(1,0,1^r,0,1) \geq 0$$

is equivalent to the inequality

$$\int_{-\infty < t_1 \leq \ldots \leq t_{r+4} < \infty} \cdot \quad \cdot \quad \cdot \int f(t_1-\theta)f(t_2)f(t_{r+3})f(t_{r+4}-\theta)[\prod_{i=3}^{r+3} f(t_i) - \prod_{i=3}^{r+2} f(t_i-\theta)]dt_1\ldots dt_{r+4} \geq 0.$$

Integration of the above with respect to all the variables but t_2 and t_{r+3} (call them x and y, respectively), yields the equivalent inequality

$$\int_{-\infty}^{\infty}\int_{x}^{\infty} F(x-\theta)F(\theta-y)\{[F(y)-F(x)]^r-[F(y-\theta)-F(x-\theta)]^r\}f(x)f(y)dydx \geq 0.$$

If we transform the integral by letting $x = x'$ and $y = x'+w$ and drop the primes we get the inequality

$$(3.5) \qquad \int_0^\infty \int_{-\infty}^\infty F(x-\theta)F(\theta-x-w)\{[F(x+w)-F(x)]^r-[F(x+w-\theta)-F(x-\theta)]^r\}f(x+w)f(x)\,dx\,dw \geq 0 .$$

Let $I(w)$ denote the inner integral. If $I(w) \geq 0$ for all $w \geq 0$, then it is clearly so that $\int_0^\infty I(w)dw \geq 0$, therefore a sufficient condition for (3.5) (hence for the conclusion of the theorem) is that $I(w) \geq 0$ for $w \geq 0$.

Let $S(w, x)$ be the integrand in (3.5). Then

$$I(w) = \int_{(\theta-w)/2}^\infty S(w, x)dx + \int_{-\infty}^{(\theta-w)/2} S(w, x)dx .$$

In the second integral let $x' = \theta-x-w$; the result, after omitting primes and combining the two integrals, is

$$I(w) = \int_{(\theta-w)/2}^\infty F(x)F(\theta-x-w)f(x+w)f(x-\theta)\{[F(x+w-\theta)-F(x-\theta)]^r$$

$$- [F(x+w)-F(x)]^r\}\left[\frac{F(-x-w)f(x+w-\theta)}{f(x+w)F(\theta-x-w)} - \frac{F(x-\theta)f(x)}{f(x-\theta)F(x)}\right]dx .$$

Clearly, a sufficient condition for $I(w)$ to be non-negative for $w \geq 0$ is that the integrand above, call it $T(x, w)$, is non-negative for $w \geq 0$ and $x \geq (\theta-w)/2$.

The expression in braces in $T(x, w)$ is non-negative if and only if

$$[F(x+w-\theta)-F(x-\theta)]-[F(x+w)-F(x)] \geq 0 ,$$

for $w \geq 0$ and $x \geq (\theta-w)/2$. This is clearly so since the left member of the

inequality is the difference of the probability contents of two intervals one of which is more central than the other.

By the corollary to Lemma 1 of Appendix I, the term in square brackets is non-negative for $x \geqq (\theta-w)/2$ and $w \geqq 0$. Therefore $T(x, w) \geqq 0$.

<u>Theorem 6:</u>

If N holds and $\theta \neq 0$, then $P(0^r,0,1,1,0,0^r) > P(0^r,1,0,0,1,0^r)$ or, equivalently, $P(1^r,1',0,0,1',1^r) > P(0^r,1',0,0,1',0^r)$.

<u>Proof:</u>

Proceeding as was outlined above, one obtains, as a sufficient condition for the inequality $P(1^r,1',0,0,1',1^r) \geqq P(0^r,1',0,0,1',0^r)$, the inequality

$$T(x, w) = \frac{(2r+2)!}{2(r!)} f(x-\theta)f(x+w-\theta)f(x)f(x+w)[G^2(x; w) - G^2(x-\theta; w)]\{[F(x-\theta)F(\theta-x-w)]^r-[F(x)F(-x-w)]^r\} \geqq 0,$$

for $\theta \geqq 0$, $w \geqq 0$, and $x \geqq (\theta-w)/2$, where

$$G(x; w) = \frac{1}{\sqrt{2\pi}} \frac{F(x+w)-F(x)}{f(\frac{x+w}{\sqrt{2}})f(\frac{x}{\sqrt{2}})} . \tag{3.6}$$

By the corollary to Lemma 2 of Appendix I, with $r = w/2$, $y = x-r$, the term in square brackets is non-negative for $\theta \geqq 0$, $w \geqq 0$, and $x \geqq (\theta-w)/2$.

Therefore $T(x; w)$ is non-negative provided the term in braces is non-negative. This term is non-negative if and only if

$$[F(x-\theta)F(\theta-x-w)-F(x)F(-x-w)] \geqq 0, \qquad \text{for} \quad x \geqq (\theta-w)/2 .$$

This inequality is proved in the corollary to Lemma 3 of Section 5.

<u>Theorem 7:</u>

If N holds and $\theta \geqq 0$, then $P(0,1,1,0,z) \geqq P(1,0,0,1,z)$ for any z or,

equivalently, $P(z,1',0,0,1') \geq P(z,0',1,1,0')$.

Proof:

Proceeding as was outlined above, one obtains as a sufficient condition for the inequality $P(z,0,1,1,0) \geq P(z,1,0,0,1)$, the inequality

$$T(x, w) = [H(x)-H(\theta-w-x)]\{G^2(x; w)-G^2(x-\theta; w)\}f(x)f(x+w)f(x-\theta)f(x+w-\theta) \geq 0,$$

for $w \geq 0$ and $x \geq (\theta-w)/2$, where $G(x; w)$ is defined by (3.6) and

$$H(x) = \frac{(m+2)!(n+2)!}{2} \int \cdot \cdot \cdot \int_{-\infty < t_1 \leq \ldots \leq t_{m+n} < x} \prod_{i=1}^{m+n} f(t_i-\theta z_i)dt_i ,$$

m and n being the number of zeros and ones in z.

It is shown by the Corollary to Lemma 2 of Appendix I that the term in braces in $T(x; w)$ is non-negative for $w \geq 0$ and $x \geq (\theta-w)/2$.

Clearly, $H(x)$ is everywhere non-decreasing. Since $x \geq (\theta-w)/2$ implies $x \geq \theta-w-x$, we have

$$H(x) \geq H(\theta-w-x) ,$$

for $w \geq 0$, $x \geq (\theta-w)/2$. Thus, the term in square brackets is non-negative, and, therefore, so is $T(x, w)$.

4. Examples and Conjectures.

The following diagrams illustrate the theorems of Section 3. The symbol $z \xrightarrow{abc} z'$ means $P(z) \geq P(z')$ for $\theta \geq 0$ (under ST, STU, or N) under conditions abc. C stands for Conjecture. The table accompanying some of the diagrams is extracted from an unpublished table of probabilities of rank orders under N computed by Jerome Klotz (1962). Note: MLR implies a simple ordering of the $P(z)$ for n=1. Hence the first interesting case is m=n=2. All diagrams derived

are distributive lattices but the conjectured diagram for $m=4$ and $n=2$ is not distributive; in particular it does not satisfy the Jordan-Dedekind chain condition, i.e., not all chains from an arbitrary fixed z to (say) the least probable rank order are of the same length.

The notation, crossover $\left\{\begin{matrix} z \\ z' \end{matrix}\right.$, denotes the fact that there exist two values of θ, $\theta_1 > \theta_2 \geq 0$, such that $P(z|\theta_1) > P(z'|\theta_1)$ but $P(z|\theta_2) < P(z'|\theta_2)$.

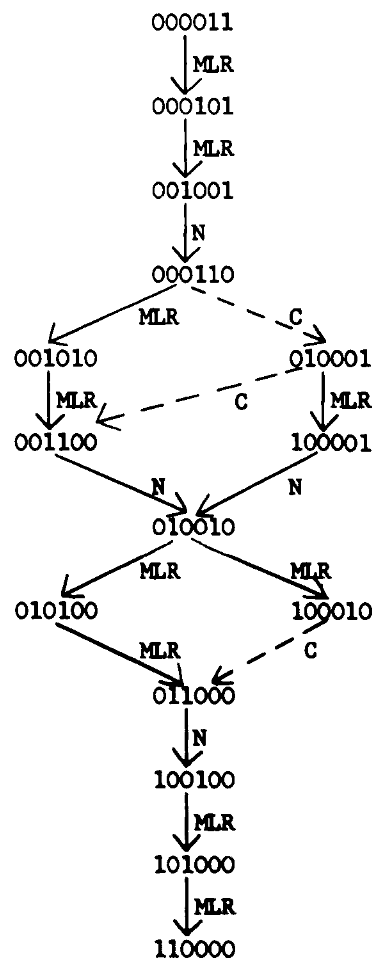
<u>m=n=2</u>

$$\begin{array}{c} 0011 \\ \downarrow \text{MLR} \\ 0101 \\ \downarrow \text{MLR} \\ 0110 \overset{\text{ST}}{=} 1001 \\ \downarrow \text{MLR} \\ 1010 \\ \downarrow \text{MLR} \\ 1100 \end{array}$$

<u>m=3, n=2</u>

$$\begin{array}{c} 00011 \\ \downarrow \text{MLR} \\ 00101 \\ \downarrow \text{MLR} \\ 01001 \\ \downarrow \text{N} \\ 00110 \\ \downarrow \text{ST and U} \\ 10001 \\ \downarrow \text{N} \\ 01010 \\ \downarrow \text{MLR} \\ 01100 \\ \downarrow \text{N} \\ 10010 \\ \downarrow \text{MLR} \\ 10100 \\ \downarrow \text{MLR} \\ 11000 \end{array}$$

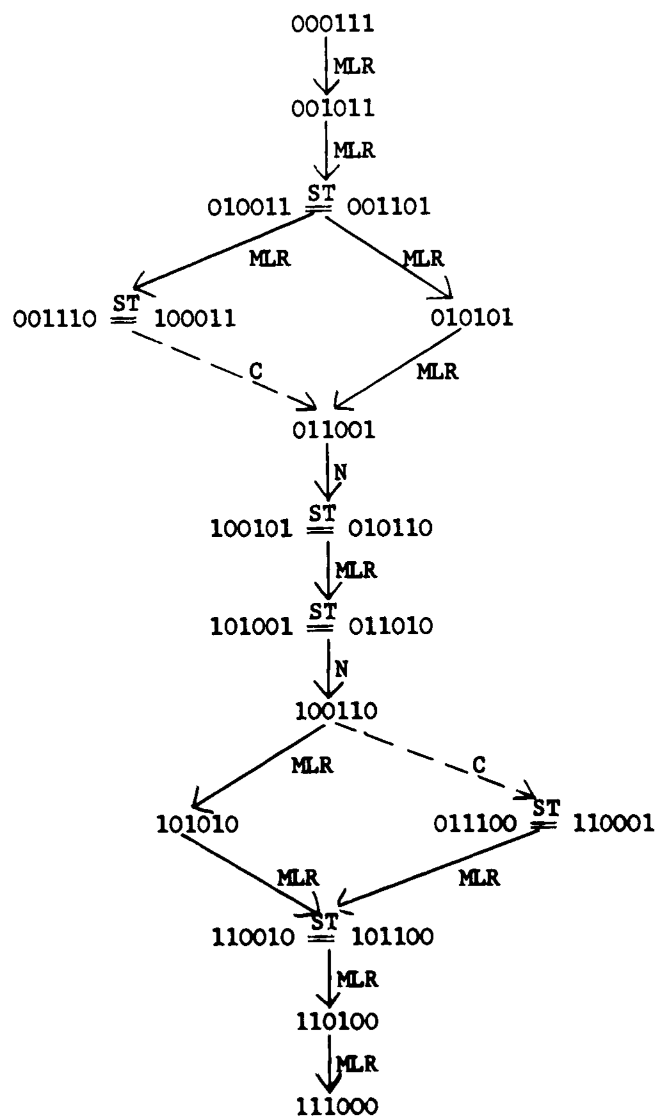
m=4, n=2



Ordering from Klotz Table

000011
000101
001001
000110
010001
001010
100001
001100
010010
010100
100010
011000
100100
101000
110000

m=n=3



Ordering from Klotz Table

000111

001011

001101 = 010011

010101 *

100011 = 001110

011001

100101 = 010110

101001 = 011010

100110

110001 = 011100

101010

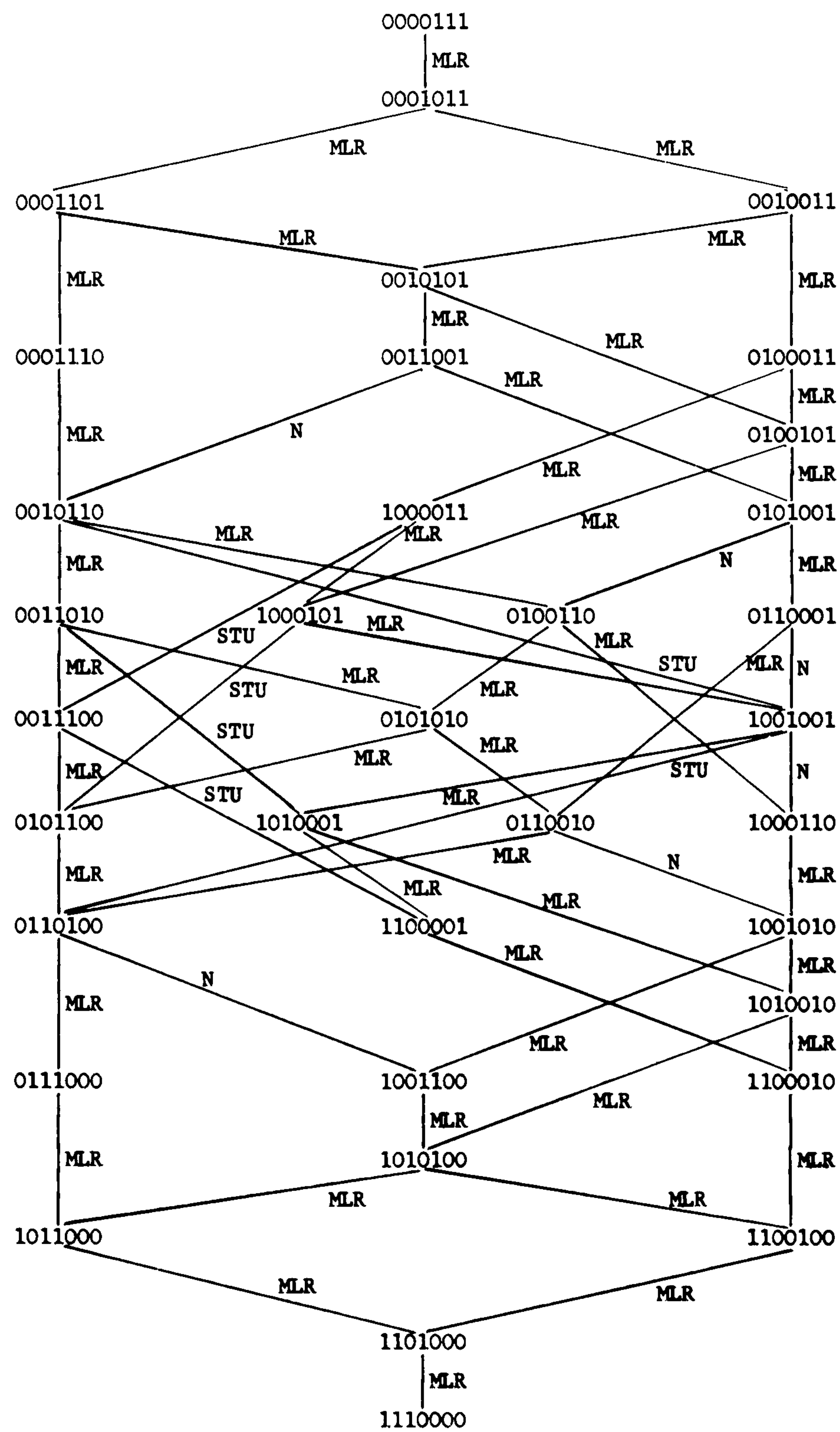
110010 = 101100

1101000

111000

* crossover

m=4, n=3


0000111
MLR
0001011
MLR
MLR
0001101
0010011
MLR
MLR
MLR
MLR
0010101
MLR
MLR
0001110
0011001
0100011
MLR
MLR
N
0100101
MLR
MLR
MLR
0010110
1000011
0101001
MLR
MLR
MLR
N
MLR
0011010
1000101
0100110
0110001
MLR
STU
MLR
MLR
STU
MLR
N
MLR
STU
MLR
0011100
0101010
1001001
STU
MLR
MLR
STU
N
MLR
MLR
STU
0101100
1010001
0110010
1000110
MLR
N
MLR
MLR
MLR
MLR
0110100
1100001
1001010
MLR
N
MLR
MLR
1010010
MLR
MLR
0111000
1001100
1100010
MLR
MLR
MLR
1010100
MLR
MLR
1011000
1100100
MLR
MLR
1101000
MLR
1110000

Ordering from Klotz Table

m=4, n=3

	0000111			0101010	
	0001011			1000110	*
	0010011			1010001	
	0001101			0110010	
	0100011			0101100	
	0010101	*		1001010	*
	0001110		*	1100001	
	0011001			0110100	
*	0100101			1010010	*
	1000011	*		1001100	*
	0010110			0111000	
	0101001			1100010	
	0011010			1010100	
	1000101			1011000	
	0100110	*		1100100	
	0110001			1101000	
	0011100			1110000	
	1001001				

* crossover

For fixed m and n we define N(z) and N'(z) to be the number of rank orders less probable and more probable than z, respectively. The ideal situation for constructing tests of hypothesis is to have $N(z)+N'(z) = \binom{m+n}{n}-1$, i.e., the rank orders form a chain. The following table gives N(z), N'(z) and N(z)+N'(z) for m=4, n=3 for the ordering implied by Theorem 3 alone and for the ordering implied by Theorems 1 through 7. Note that the second ordering is an improvement over the first in the sense that N(z)+N'(z) for the second ordering is not smaller than that for the first. In particular there is considerable improvement in z = (1,0,0,0,0,1,1), (0,0,1,0,1,1,0), (0,1,1,0,0,0,1) and (0,0,1,1,1,0,0).

z	MLR N(z)	MLR N'(z)	MLR N(z)+N'(z)	MLR or STU or N N(z)	MLR or STU or N N'(z)	MLR or STU or N N(z)+N'(z)
0000111	34	0	34	34	0	34
0001011	33	1	34	33	1	34
0010011	30	2	32	30	2	32
0001101	29	2	31	29	2	31
0100011	24	3	27	25	3	28
0010101	27	4	31	28	4	32
0001110	19	3	22	22	3	25
0011001	21	5	26	24	5	29
0100101	22	6	28	22	6	28
1000011	14	4	18	19	5	24
0010110	18	6	24	21	7	28
0101001	20	8	28	20	8	28
0011010	15	8	23	17	8	25
1000101	13	8	21	16	8	24
0100110	15	9	24	15	11	26
0110001	12	9	21	16	9	25
0011100	9	9	18	11	11	22
1001001	11	11	22	12	12	24
0101010	13	13	26	13	13	26

To complete the table note that $N(z^t) = N'(z)$.

Appendix I.

Properties of Some Functions Related to the Normal Density Function

In the following $f(\cdot)$ and $F(\cdot)$ denote the standard normal density and distribution function., respectively.

Lemma 1:

If θ is a positive constant, then $G(x) = \frac{F(x-\theta)f(x)}{F(x)f(x-\theta)}$ is non-increasing for all x.

Proof:

A sufficient condition for the monotonicity of $G(x)$ is that its first derivative is non-positive for all x, or, equivalently, that

$$G_1(x) = \theta F(x)F(x-\theta)+F(x-\theta)f(x)-F(x)f(x-\theta) \geq 0,$$

for all x.

Since $G_1(-\infty) = 0$, a sufficient condition for $G_1(x) \geq 0$ is that $G_1(x)$ is everywhere non-decreasing, which is so provided the first derivative of $G_1(x)$ is non-negative.

Now,

$$\frac{d}{dx} G_1(x) = f(x)f(x-\theta)\left[\frac{xF(x)}{f(x)} - \frac{(x-\theta)F(x-\theta)}{f(x-\theta)}\right] .$$

Let $G_2(x) = \frac{xF(x)}{f(x)}$. Clearly, the term in square brackets is non-negative (hence $\frac{d}{dx} G_1(x) \geq 0$) if $G_2(x)$ is non-decreasing for all x, that is, if

$$f^{-1}(x)[xf(x)+F(x)(x^2+1)] \geq 0 .$$

This is clearly so for $x \geq 0$. To see that it follows for $x < 0$, let $G_3(x) = [xf(x)+F(x)(x^2+1)]$. Since $G_3(-\infty) = 0$ we need, as was noted above, show only that $G_3(x)$ is non-decreasing for all x, or that $\frac{d}{dx} G_3(x) = 2[xF(x)+f(x)] \geq 0$, which follows at once from the Feller-Laplace inequality:

$$F(x) \leq \frac{F(x)}{-x} \qquad \text{for } x < 0 .$$

Corollary:

If w and θ are non-negative constants, then $G(\theta-x-w) \geq G(x)$ for $x \geq (\theta-w)/2$.

Proof:

The conclusion follows at once from the fact that $\theta-x-w \leq x$ whenever $x \geq (\theta-w)/2$.

Lemma 2:

If r is a positive constant, then $H(y, r) = \dfrac{F(y+r)-F(y-r)}{f(\frac{y+r}{\sqrt{2}})f(\frac{y-r}{\sqrt{2}})}$ is non-decreasing for $y \geqq 0$.

Proof:

A sufficient condition for $H(y, r)$ to be non-decreasing for $y \geqq 0$ is that its first derivative is non-negative, or, equivalently, that

$$H_1(y, r) = f(y+r)-f(y-r)+y[F(y+r)-F(y-r)] \geqq 0 .$$

Since $H_1(y, 0) = 0$ it is sufficient to show that $H_1(y, r)$ is increasing in r for $r \geqq 0$ and fixed $y \geqq 0$, or that

$$\frac{d}{dr} H_1(y, r) = r[f(y-r)-f(y+r)] \geqq 0 ,$$

for $y \geqq 0$ and $r \geqq 0$, which is clearly so.

Corollary:

Let θ be a positive constant, then $[H^2(y; r)-H^2(y-\theta; r)] \geqq 0$ for all $y \geqq \theta/2$ and $r \geqq 0$.

Proof:

It is evident that $H(y; r)$ is symmetric about $y=0$ (for fixed $r \geqq 0$), therefore $H(y-\theta; r)$ is symmetric about θ. In the interval $[\theta/2, \theta]$, $H(y-\theta; r)$ is decreasing while $H(y; r)$ is increasing. Since the two are equal at $y = \theta/2$, clearly, $H(y-\theta; r)$ is less than or equal to $H(y; r)$ in this interval. Since $H(y-\theta; r)$ is increasing to the right of this point and, $y \geqq y-\theta$, one has $H(y; r) \geqq H(y-\theta; r)$. Since $H(\cdot)$ is non-negative, the result follows at once.

Lemma 3:

Let r be a positive constant, then $F(y-r)F(-y-r)$ is non-increasing in y for $y \geqq 0$.

Proof:

It is enough to show that the first derivative is non-positive, or, equivalently, that

$$\frac{f(y+r)}{F(y+r)} \leq \frac{f(r-y)}{F(r-y)} ,$$

for $r \leq 0$, $y \geqq 0$.

Since equality holds for $y=0$ it is enough to show that $\frac{f(t)}{F(t)}$ is non-increasing for all t, or that $-tF(t)-f(t) \leq 0$, for all t. If $t \geqq 0$ this is clear, if $t < 0$ apply the Feller-Laplace inequality

$$F(t) \leq \frac{f(t)}{-t} .$$

Corollary:

If θ and w are non-negative constants, then

$$F(x-\theta)F(\theta-x-w)-F(x)F(-x-w) \geqq 0$$

for $x \geqq (\theta-w)/2$.

Proof:

Let $H(y;\ r) = -F(y-r)F(-y-r)$, using this notation we are to show that

$$H(y;\ r)-H(y-\theta;\ r) \geqq 0$$

for $y \geqq \theta/2$ and $r \geqq 0$ where $H(y;\ r)$ is non-decreasing for $y \geqq 0$ and symmetric about $y=0$. This is proved as in the corollary to Lemma 2.

Appendix II.

Selected Numerical Results

The following tables give for several z's, values of $P(z)$ for different values of θ under condition N. These tables were extracted from an unpublished

table of Klotz. Attention is directed towards z = (1,0,0,0,0,1,0,1) and z' = (0,0,1,0,1,0,1,0). This was the first example of a pair of rank orders with common m and n values for which we have found the function $P(z|\theta)-P(z'|\theta)$ to change sign on the positive θ axis. The second example of such a "crossover"* on the positive θ axis that we found was with the vectors z = (0,1,0,1,0,1) and z' = (0,0,1,1,1,0) [or its equivalent by ST, z'' = (1,0,0,0,1,1)], where m=n=3. An exhaustive search has not been made of the Klotz tables.

*crossover defined on page 12.

P(z) for selected values of θ under condition N when m=3 and n=2 (from Klotz)

z \ θ	.25	.50	1.0	1.5	2.0	3.0	4.0	5.0	6.0
00011	.14772	.20814	.36243	.54014	.70717	.92051	.98746	.99883	.99993
00101	.12978	.15869	.19948	.20069	.16348	.06050	.01115	.00112	$.0_4 6$
01001	.11445	.12292	.11764	.08837	.05260	.00968	.00080	$.0_4 3$	$.0_6 64$
00110	.10975	.11333	.10088	.07097	.03976	.00655	.00049	$.0_4 2$	$.0_6 34$
10001	.09706	.08874	.06209	.03439	.01517	.00154	$.0_4 7$	$.0_5 153$	$.0_7 1$
01010	.09669	.08741	.05838	.02982	.01166	.00081	$.0_4 2$	$.0_6 17$	0
01100	.08568	.06899	.03710	.01552	.00505	.00025	$.0_5 463$	$.0_7 3$	0
10010	.08193	.06290	.03041	.01125	.00317	.00011	$.0_5 125$	0	0
10100	.07253	.04946	.01905	.00567	.00130	$.0_4 3$	$.0_6 23$	0	0
11000	.06443	.03942	.01255	.00318	.00064	$.0_4 1$	$.0_7 7$	0	0

Subscripts on the first zero after the decimal indicate the number of zeros to be entered; for example, $.0_4 6$ stands for .00006.

P(z) for selected values of θ under condition N when m=4 and n=2 (from Klotz)

z \ θ	.25	.50	1.0	1.5	2.0	3.0	4.0	5.0	6.0
000011	.10454	.15548	.29662	.47430	.65377	.90079	.98360	.99847	.999914
000101	.09313	.12192	.17290	.19240	.17020	.07084	.01398	.00145	$.0_4 85$
001001	.08400	.09871	.11104	.09580	.06412	.01411	.00132	$.0_4 57$	$.0_5 121$
000110	.07955	.08871	.09032	.07094	.04340	.00804	.00064	$.0_4 24$	$.0_6 46$
010001	.07521	.07903	.07080	.04837	.02547	.00341	.00019	$.0_5 47$	0
001010	.07170	.07158	.05712	.03397	.01510	.00127	$.0_4 35$*	$.0_6 32$*	0
100001	.06450	.05843	.03941	.02055	.00835	.00068	$.0_4 23$*	$.0_6 37$*	0
001100	.06433	.05781	.03771	.01853	.00688	.00041	$.0_5 8$	$.0_7 6$	0
010010	.06415	.05717	.03607	.01674	.00572	.00027	$.0_5 4$	$.0_7 1$	0
010100	.05753	.04606	.02357	.00892	.00250	$.0_4 79$	$.0_6 75$	0	0
100010	.05500	.04219	.01992	.00698	.00181	$.0_4 48$	$.0_6 38$	0	0
011000	.05317	.03810	.01647	.00537	.00131	$.0_4 33$	$.0_6 26$	0	0
100100	.04929	.03392	.01291	.00366	.00077	$.0_4 13$	$.0_7 7$	0	0
101000	.04467	.02798	.00893	.00215	.00039	$.0_5 53$	$.0_7 1$	0	0
110000	.04022	.02290	.00620	.00130	.00021	$.0_5 23$	$.0_7 1$	0	0

Subscripts on the first zero after the decimal indicate the number of zeros to be entered; for example, $.0_4 85$ stands for .000085.

*This appears to be another crossover but it is not clear that the eighth decimal should be trusted in this calculation.

P(z) for selected values of θ under condition N when m=n=3 (from Klotz)

z \ θ	.25	.50	1.0	1.5	2.0	3.0	4.0	5.0	6.0
000111	.08222	.12748	.26025	.43721	.62357	.88989	.98186	.99827	.99990
001011	.07392	.10181	.15726	.18692	.17369	.07673	.01554	.00162	.00010
001101	.06603841	.08081	.09679	.08695	.05941	.01301	.00116	$.0_4 5$	$.0_6 97$
010011	**same as**	001101							
010101	.05896	.06396	.05878	.03906	.01889*	.00176*	$.0_4 5$	$.0_6 48$	0
001110	.05654021	.05935	.05249	.03492	.01768*	.00212*	.00010	$.0_5 23$	$.0_7 2$
100011	**same as**	001110							
011001	.05335	.05244	.03974	.02190	.00884	.00059	$.0_4 1255$	$.0_7 9$	0
010110	.05045053	.04686	.03156	.01530	.00535	.00025	$.0_5 334$	$.0_7 1$	0
100101	**same as**	010110							
011010	.04561	.03830	.02108	.00835	.00238	$.0_4 7425$	$.0_6 66$	0	0
101001	**same as**	011010							
100110	.04315	.03427	.01682	.00589	.00147	$.0_4 3305$	$.0_6 19$**	0	0
011100	.04101	.03118	.01433	.00485	.00120	$.0_4 3$	$.0_6 22$**	0	0
110001	**same as**	011100							
101010	.03898	.02792	.01109	.00312	.00062	$.0_5 842$	$.0_7 2$	0	0
101100	.03503	.02268	.00748	.00178	.00030	$.0_5 317$	$.0_7 1$	0	0
110010	**same as**	101100							
110100	.03146	.01839	.00501	.00100	.00015	$.0_5 115$	0	0	0
111000	.02859	.01534	.00361	.00064	$.0_4 84$	$.0_6 57$	0	0	0

Subscripts on the first zero after the decimal indicate the number of zeros to be entered; for example, $.0_4 5$ stands for .00005.

***This appears to be a bona-fide crossover.**

****This crossover may have resulted from a computational error.**

P(z) for selected values of θ illustrating a "crossover" and "symmetrical displacements" (from Klotz)

z \ θ	.25	.50	1.0	1.5	2.0	3.0	4.0	5.0	6.0
Example of a crossing over.									
10000101	.01911	.01851	.01285461	.00601	.00190	$.0_4 6$	$.0_6 44$	0	0
00101010	.01933	.01878	.01285266	.00574	.00168	$.0_4 4$	$.0_6 19$	0	0
Effect of moving in from both ends.									
10000001	.03442	.03082	.01984	.00958	.00351	.00022	$.0_5 516$	$.0_7 5$	0
01000010	.03419	.03000	.01778	.00744	.00220	$.0_4 6815$	$.0_6 53$	0	0
00100100	.03418	.02997	.01771	.00737	.00216	$.0_4 6541$	$.0_6 49$	0	0
00011000	.03426	.03025	.01841	.00804	.00253	$.0_4 9$	$.0_6 96$	0	0

Subscripts on the first zero after the decimal indicate the number of zeros to be entered; for example $.0_4 6$ stands for .00006.

Appendix III.

Approximations to $P(z|\theta)$ under N

Theorem 8:

If N holds, then, for all θ,

$$P(z) = \binom{m+n}{n}^{-1} e^{-n\theta^2/2} \, E \exp\{\theta \sum_{i=1}^{m+n} z_i W_i\} ,$$

where $w_1, \ldots, w_{m+n}$ are the order statistics of a sample of size m+n drawn from a standard normal population.

Proof:

We note that in this case

$$f(t_i - z_i\theta) = e^{-z_i\theta^2/2} e^{z_i t_i \theta} f(t_i) .$$

Thus

$$P(\mathbf{z}) = m!n! \int_{-\infty < t_1 \leq \ldots \leq t_{m+n} < \infty} \cdot \quad \cdot \quad \cdot \int \prod_{i=1}^{m+n} f(t_i - z_i\theta)dt_i$$

$$= \binom{m+n}{n}^{-1} e^{-n\theta^2/2}(m+n)! \int_{-\infty < t_1 \leq \ldots \leq t_{m+n} < \infty} \cdot \quad \cdot \quad \cdot \int \exp\{\theta \sum_{i=1}^{m+n} z_i t_i\} \prod_{i=1}^{m+n} f(t_i)dt_i .$$

Remarks

The first terms in the Maclaurin expansion of $P(\mathbf{z}|\theta)$ give approximations to $P(\mathbf{z};\ \theta)$ for small θ. For example

$$P(\mathbf{z}|\theta) \doteq \binom{m+n}{n}^{-1}(1 + \theta \sum_{i=1}^{m+n} z_i EW_i) .$$

Tables of the approximation got by including the next (θ^2) term in the Maclaurin expansion of $P(\mathbf{z}|\theta)$ are in preparation.

An interesting asymptotic result is given by Hodges and Lehmann (1962). This should give approximations to $P(\mathbf{z}|\theta)$ for large θ; this point has not been investigated, however.

Appendix IV.

Some Nonlinear Relationships Between Rank Order Probabilities

In Savage (1960)p. 520, linear relationships like the following have been obtained:

$$P(0,1,1) = [P(0,1,1,0) + P(0,1,0,1) + 2P(0,0,1,1)]/2 .$$

A pair of non-linear relationships is obtained below. Note that no restriction is made of the densities $f(\cdot)$ and $g(\cdot)$.

Theorem 9:

$$A: \quad P(0,1,1,0) = 2P(0,1,1) - 2P^2(0,1)$$

$$B: \quad P(0,1,0,1) = 2P^2(0,1) - 2P(0,0,1,1) \ .$$

Proof:

We note first that

$$P^2(0,1) = [\int_{-\infty}^{\infty} F(x)g(x)dx]^2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x)g(x)F(y)g(y)dxdy$$

$$= 2\int_{-\infty}^{\infty}\int_{x}^{\infty} F(x)F(y)g(x)g(y)dydx \ .$$

Then to prove A, one has

$$P(0,1,1,0) = 4\int_{-\infty}^{\infty}\int_{x}^{\infty} F(x)[1-F(y)]g(x)g(y)dydx$$

$$= 4\int_{-\infty}^{\infty}\int_{x}^{\infty} F(x)g(x)g(y)dydx - 2P^2(0,1)$$

$$= 2P(0,1,1) - 2P^2(0,1) \ .$$

And to prove B, one has

$$P(0,1,0,1) = 4\int_{-\infty}^{\infty}\int_{x}^{\infty} F(x)[F(y)-F(x)]g(x)g(y)dydx$$

$$= 2P^2(0,1) - 4\int_{-\infty}^{\infty}\int_{x}^{\infty} F^2(x)g(x)g(y)dydx = 2P^2(0,1) - 2P(0,0,1,1).$$

Corollary 1:

$$A': \quad P(1,0,0,1) = 2P(1,0,0) - 2P^2(1,0)$$

$$B': \quad P(1,0,1,0) = 2P^2(1,0) - 2P(1,1,0,0)$$

Proof:

Simply interchange F and G (f and g) in the proofs of A and B.

Corollary 2:

$$P(0,0,1,1) + P(1,1,0,0) = 2[P(0,1,1) + P(1,0,0)] - 1 .$$

Proof:

The set of all possible rank orders for $m=n=2$ is an exhaustive set of mutually exclusive events. Therefore

$$P(0,0,1,1)+P(1,1,0,0)+P(0,1,0,1)+P(1,0,1,0)+P(1,0,0,1)+P(0,1,1,0) = 1 .$$

Substituting the right members of A, B, A' and B' for P(0,1,1,0), P(0,1,0,1), P(1,0,0,1) and P(1,0,1,0), we obtain

$$P(0,0,1,1) + P(1,1,0,0) = 2[P(0,1,1) + P(1,0,0)] - 1 .$$

Note that probabilities for all of the rank orders with $m=n=2$ can be evaluated in terms of the probabilities for smaller sample sizes and P(0,0,1,1) or P(1,1,0,0). More generally, for $n=2$ and arbitrary fixed $m=M$ let

$$z = (\underbrace{0,\ldots,0}_{r_1},\ 1,\ \underbrace{0,\ldots,0}_{r_2},\ 1,\ \underbrace{0,\ldots,0}_{r_3}) ,$$

where $r_1+r_2+r_3 = M$. Then

$$P(z) = \frac{M!2!}{r_1!r_2!r_3!} \int \cdots \int_{-\infty < x \leq y < \infty} F^{r_1}(x)[F(y)-F(x)]^{r_2}[1-F(y)]^{r_3} g(x)g(y)dxdy$$

$$= \frac{M!2!}{r_1!r_2!r_3!} \sum_{\substack{1\leq i_1\leq r_2 \\ 1\leq i_2\leq r_3}} \binom{r_2}{i_1}\binom{r_3}{i_2}(-1)^{r_2+r_3-(i_1+i_2)} \int \cdots \int_{-\infty < x \leq y < \infty} F^{r_1+r_2-i_1}(x)\cdot F^{r_3+i_1-i_2}(y)g(x)g(y)dxdy .$$

For $a+b < M$, the integral

$$\int\cdots\int_{-\infty < x \leq y < \infty} F^a(x)F^b(y)g(x)g(y)dxdy$$

can be expressed as a linear combination of probabilities of rank orders for $m < M$ and $n=1$ or 2. Therefore if all rank order probabilities for $m < M$ and $n=1$ and 2 have been computed, the only new integrals required are of the form

$$A_i = \int\cdots\int_{-\infty < x \leq y < \infty} F^i(x)F^{M-i}(y)g(x)g(y)dxdy , \qquad i=0,\ldots,M .$$

Since

$$A_i+A_{M-i} = \int\cdots\int_{-\infty < x \leq y < \infty} [F^i(x)F^{M-i}(y)+F^{M-i}(x)F^i(y)]g(x)g(y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^i(x)F^{M-i}(y)g(x)g(y)dxdy$$

$$= [\int_{-\infty}^{\infty} F^i(x)g(x)dx]\ [\int_{-\infty}^{\infty} F^{M-i}(y)g(y)dy]$$

one needs to compute only one of the pair (A_i, A_{M-i}).

For $n > 2$ we must consider n-fold integrals of the form

$$A(i_1,\ldots,i_n) = \int\cdots\int_{-\infty < x_1 \leq\ldots\leq x_n < \infty} \prod_{j=1}^{n} F^{i_j}(x_j)g(x_j)dx_j ,$$

as above only those integrals for which $\sum_{j=1}^{n} i_j = M$ need be evaluated. Then if any of the $i_j=0$ the dimensionality of the integral is easily decreased.

Generally,

$$\Sigma A(i_1,\ldots,i_n) = \prod_{j=1}^{n} P(\underbrace{0,\ldots,0}_{i_j}, 1)$$

where the summation is over all permutations of $(1,2,\ldots,n)$.

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